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Weak Continuity Between *WSS* and *GTS* due to Császár

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ABSTRACT

In this paper, we define and investigate the notions of weak (μ, w)-continuity and weak (w, μ)-continuity between a weak structure space (Császár (2011)) and a generalized topological space (Császár (2002)). Also, we obtain several characterizations and many properties of these notions.

Keywords: Weak structure, generalized topology, weak (w, μ)-continuity, weak (μ, w)-continuity.

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1. INTRODUCTION AND PRELIMINARIES

Császár (2002) introduced a generalized structure called generalized topology. Recently, Császár (2011) has introduced a new notion of structures called a weak structure which is weaker than both a generalized topology (Császár (2002)) and a minimal structure (Popa and Noiri (2000)). Let X be a nonempty set and $w \subseteq P(X)$, where $P(X)$ is the power set of X. Then w is called a weak structure (briefly WS) on X if $\phi \in w$. A nonempty set X with a weak structure is called a weak structure space (briefly WSS) and is denoted by (X, w). Each member of

w is said to be w-open and the complement of a w-open set is said to be w-closed. Let w be a weak structure on X and $A \subseteq X$. Császár (2011) defined (as in the general case) $i_w(A)$ as the union of all w-open subsets of A (e.g. ϕ) and $c_w(A)$ as the intersection of all w-closed sets containing A (e.g. X), for getting a deeper insight into further studying some topological notions of different kinds see Agnello and Cammaroto (1987) for details.

We call a class $\mu \subseteq P(X)$ a generalized topology (Császár (2002)) (briefly GT) if $\phi \in \mu$ and the arbitrary union of elements of μ belongs to μ. A set X with a GT μ on it is called a generalized topological space (briefly GTS) and is denoted by (X, μ). Quite recently, Al-Omari and Noiri have introduced the notions of (w,k)-continuity and weak (w,k)-continuity between weak structure spaces and obtained several characterizations and a number of properties of the notions. In this paper, we define and investigate the notions of weak (μ, w)-continuity and weak (w, μ)-continuity between a weak structure space Császár (2011) and a generalized topological space Császár (2002). Also, we obtain several characterizations and many properties of these notions.

The following lemmas are useful in the sequel:

Lemma 1.1 (Császár (2011)). *Let w be a WS on X and A, B subsets of X, then the following properties hold*

1. $i_w(A) \subseteq A \subseteq c_w(A)$.
2. If $A \subseteq B$ implies that $i_w(A) \subseteq i_w(B)$ and $c_w(A) \subseteq c_w(B)$.
3. $i_w(i_w(A)) = i_w(A)$ and $c_w(c_w(A)) = c_w(A)$.
4. $i_w(X - A) = X - c_w(A)$ and $c_w(X - A) = X - i_w(A)$.

Lemma 1.2 (Császár (2011)). *Let w be a WS on X and A a subset of X, then the following properties hold*

1. $x \in i_w(A)$ if and only if there is $W \in w$ such that $x \in W \subseteq A$.
2. $x \in c_w(A)$ if and only if $W \cap A \neq \phi$ whenever $x \in W \in w$.
3. If $A \in w$, then $A = i_w(A)$ and if A is w-closed, then $A = c_w(A)$.

Remark 1.3 *If w is a WS on X, then*

1. $i_w(\phi)=\phi$ and $c_w(X)=X$.
2. $i_w(X)$ is the union of all w-open sets in X.
3. $c_w(\phi)$ is the intersection of all w-closed sets in X.

As in (Császár (2008)), let μ be a GT on a nonempty set X and $P(X)$ the power set of X. Let us define the collection $\theta(\mu)\subseteq P(X)$ as follows: $A\in\theta(\mu)$ if and only if for each $x\in A$, there exists $U\in\mu$ such that $x\in U\subseteq c_\mu(U)\subseteq A$. Then $\theta(\mu)$ is also a GT included in μ. The element of $\theta(\mu)$ is called a $\theta(\mu)$-open set and the complement is called $\theta(\mu)$-closed. Let (X,μ) be a GTS and $A\subseteq X$. We recall the the notions defined as follows:

1. $\gamma_\theta(A)=\{x\in X : c_\mu(U)\cap A\neq\phi$ for all μ-open set U containing $x\}$. (Császár (2002))
2. $c_\theta(A)=\cap\{F\subseteq X : A\subseteq F, F$ is $\theta(\mu)$-closed in $X\}$. (Min (2009))
3. $i_\theta(A)=\cup\{V\subseteq X : V\subseteq A, V$ is $\theta(\mu)$-open in $X\}$. (Min (2009))
4 $\iota_\theta(A)=\{x\in X : c_\mu(U)\subseteq A$ for some μ-open set U containing $x\}$. (Min (2009))

Theorem 1.4 (Császár (2008)). *Let (X,μ) be a GTS and $A\subseteq X$. Then the following hold*

1. $i_\theta(A)\subseteq\iota_\theta(A)\subseteq i_\mu(A)\subseteq A\subseteq c_\mu(A)\subseteq\gamma_\theta(A)\subseteq c_\theta(A)$.
2. If A is μ-open, then $c_\mu(A)=\gamma_\theta(A)$.

Lemma 1.5 *Let (X,μ) be a GTS and $A\subseteq X$. Then $\gamma_\theta(A)$ is μ-closed.*

Proof. Let $x\in X-\gamma_\theta(A)$, then $x\notin\gamma_\theta(A)$. There exists $U_x\in\mu$ containing x such that $c_\mu(U_x)\cap A=\phi$. Then we have $U_x\cap\gamma_\theta(A)=\phi$ and hence $x\in U_x\subseteq X-\gamma_\theta(A)$ and we have $X-\gamma_\theta(A)=\cup U_x\in\mu$. Therefore, $\gamma_\theta(A)$ is μ-closed.

2. (μ, w)-CONTINUOUS FUNCTIONS

Definition 2.1 *Let μ be a GT on X and w be a WS on Y. A function $f:(X,\mu)\to(Y,w)$ is said to be (μ,w)-continuous if for each $x\in X$ and each $V\in w$ containing $f(x)$, there exists $U\in\mu$ containing x such that $f(U)\subseteq V$.*

Definition 2.2 (Császár (2008)). *Let μ and λ be generalized topologies on X and Y, respectively. A function $f:(X,\mu)\to(Y,\lambda)$ is said to be (μ,λ)-continuous if $f^{-1}(V)\in\mu$ for every $V\in\lambda$.*

Definition 2.3 *Let w and k be weak structures on X and Y, respectively. A function $f:(X,w)\to(Y,k)$ is said to be (w,k)-continuous if for each $x\in X$ and each $V\in k$ containing $f(x)$, there exists $U\in w$ containing x such that $f(U)\subseteq V$.*

Theorem 2.4 *Let w and k be weak structures on X and Y, respectively. For a function $f:(X,w)\to(Y,k)$, the following properties are equivalent:*

1. f is (w,k)-continuous;
2. $f^{-1}(B)=i_w(f^{-1}(B))$ for every k-open set B in Y;
3. $f(c_w(A))\subseteq c_k(f(A))$ for every subset A in X;
4. $c_w(f^{-1}(B))\subseteq f^{-1}(c_k(B))$ for every subset B in Y;
5. $f^{-1}(i_k(B))\subseteq i_w(f^{-1}(B))$ for every subset B in Y;
6. $c_w(f^{-1}(K))=f^{-1}(K)$ for every k-closed set K in Y.

Theorem 2.5 *Let μ be a GT on X and w be a WS on Y. For a function $f:(X,\mu)\to(Y,w)$, the following properties are equivalent:*

1. f is (μ,w)-continuous;
2. $f^{-1}(B)$ is μ-open in X for every w-open set B in Y;
3. $f(c_\mu(A))\subseteq c_w(f(A))$ for every subset A in X;
4. $c_\mu(f^{-1}(B))\subseteq f^{-1}(c_w(B))$ for every subset B in Y;
5. $f^{-1}(i_w(B))\subseteq i_\mu(f^{-1}(B))$ for every subset B in Y;
6. $f^{-1}(K)$ is μ-closed in X for every w-closed set K in Y.

Proof. This follows from Theorem 2.4 and the fact that A is μ-open if and only if $A = i_\mu(A)$ for $A \subseteq X$.

Let w be a WS on X and $w^* = \{A \subset X : A = i_w(A)\}$, then it is shown that

1. w^* is a GT containing w;
2. w is a GT if and only if $w = w^*$.

Theorem 2.6 *Let μ be a GT on X and w be a WS on Y. For a function $f:(X,\mu) \to (Y,w)$, the following properties are equivalent:*

1. f is (μ, w)-continuous;
2. for each $x \in X$ and each $V \in w^*$ containing $f(x)$, there exists $U \in \mu$ containing x such that $f(U) \subseteq V$;
3. $f:(X,\mu) \to (Y,w^*)$ is (μ, w^*)-continuous (in the sense of Császár (2002)).

Proof. (1) $\Rightarrow$ (2). Let $x \in X$ and $V \in w^*$ containing $f(x)$. Since $V \in w^*$, then $V = i_w(V)$ and $f(x) \in i_w(V)$. By Lemma 1.2, there exists $W \in w$ such that $f(x) \in W \subseteq V$. By (1), there exists $U \in \mu$ containing x such that $f(U) \subseteq W \subseteq V$.

(2) $\Rightarrow$ (3). Suppose that $V \in w^*$. Let $x \in f^{-1}(V)$, then $f(x) \in V = i_w(V)$. By Lemma 1.2, there exists $W \in w \subseteq w^*$ such that $f(x) \in W \subseteq V$. Then by (2) there exists $U_x \in \mu$ containing x such that $f(U_x) \subseteq W \subseteq V$. Hence $x \in U_x \subseteq f^{-1}(V)$ and hence $f^{-1}(V) = \cup\{U_x : x \in f^{-1}(V)\} \in \mu$. Therefore, $f:(X,\mu) \to (Y,w^*)$ is (μ, w^*)-continuous.

(3) $\Rightarrow$ (1). Let $x \in X$ and each V be a w-open set containing $f(x)$. Since $w \subseteq w^*$, by (3) $x \in f^{-1}(V) \in \mu$. Thus there exists $U = f^{-1}(V) \in \mu$ containing x such that $f(U) \subseteq V$. Therefore, f is (μ, w)-continuous.

3. WEAKLY (μ, w)-CONTINUOUS FUNCTIONS

Definition 3.1 *Let μ be a GT on X and w be a WS on Y. A function $f:(X,\mu)\to(Y,w)$ is said to be weakly (μ,w)-continuous at $x\in X$ if for each $V\in w$ containing $f(x)$, there exists $U\in\mu$ containing x such that $f(U)\subseteq c_w(V)$. A function $f:(X,\mu)\to(Y,w)$ is said to be weakly (μ,w)-continuous if it has the property at each point $x\in X$.*

Example 3.2 *Let $X=\{a,b,c,d\}$. Consider the generalized topology $\mu=\{\phi,\{a\},\{a,b\},\{b,c\},\{a,b,c\}\}$ and a weak structure $w=\{\phi,\{a,c\},\{b,c\}\}$ on X. Define $f:(X,\mu)\to(Y,w)$ as follows*

$$f(\{a\})=b,\ f(\{b\})=f(\{d\})=d,\ \ f(\{c\})=c.$$

Then since $c_w(\{a,c\})=c_w(\{b,c\})=X$ it is obvious that f is weakly (μ,w)-continuous. But f is not (μ,w)-continuous.

Definition 3.3 *Let X be a nonempty set which has a weak structure w and A a subset of X. The w-frontier of A, denoted by $w\text{-}Fr(A)$, is defined by $w\text{-}Fr(A)=c_w(A)\cap c_w(X-A)=c_w(A)-i_w(A)$.*

Definition 3.4 *Let μ be a GT on X and w be a WS on Y. A function $f:(X,\mu)\to(Y,w)$ is said to be weakly* (μ,w)-continuous if $f^{-1}(w-Fr(V))$ is μ-closed in X for each w-open set V of Y.*

By the two examples stated below, we show that weakly (μ,w)-continuous and weakly* (μ,w)-continuous are independent of each other.

Example 3.5 *Let $X=\{a,b,c,d\}$. Consider the generalized topology $\mu=\{\phi,\{a\},\{a,c\},\{a,b\},\{b,c\},\{a,b,c\},\{b,c,d\},X\}$ and a weak structure $w=\{\phi,\{a,c\},\{b,c\}\}$ on X. Define $f:(X,\mu)\to(X,w)$ as follows: $f(\{a\})=b,\ f(\{b\})=f(\{d\})=d,\ f(\{c\})=c$. Then, since $c_w(\{a,c\})=c_w(\{b,c\})=X$ it is obvious that f is weakly (μ,w)-continuous. But f is not weakly* (μ,w)-continuous.*

Example 3.6 *Let* $X=\{a,b,c,d\}$. *Consider the generalized topology* $\mu=\{\phi,\{b,c,d\}\}$ *and a weak structure* $w=\{\phi,\{a,c\},\{d\}\}$ *on* X. *Define* $f:(X,\mu)\to(Y,w)$ *as follows:* $f(\{a\})=b$, $f(\{b\})=f(\{d\})=d$, $f(\{c\})=c$. *Then since* $w\text{-}Fr(\{a,c\})=w\text{-}Fr(\{d\})=\{b\}$ *it is obvious that* f *is weakly** (μ,w)*-continuous. But* f *is not weakly* (μ,w)*-continuous.*

Theorem 3.7 *Let* μ *be a GT on* X *and* w *be a WS on* Y. *If a function* $f:(X,\mu)\to(Y,w)$ *is* (μ,w)*-continuous, then it is weakly* (μ,w)*-continuous and weakly** (μ,w)*-continuous. The converse is true if* μ *is a topology.*

Proof. Suppose that f is (μ,w)-continuous. It is obvious that f is weakly (μ,w)-continuous. We show that f is weakly* (μ,w)-continuous. Let V be a w-open set. Since

$$w\text{-}Fr(V)=c_w(V)\cap c_w(X-V)=c_w(V)\cap(X-V),$$

$$f^{-1}(w\text{-}Fr(V))=f^{-1}(c_w(V))\cap f^{-1}(X-V) \text{ and}$$

$$f^{-1}(c_w(V))=f^{-1}\left(\cap\{(Y-G):G\in w,V\subseteq(Y-G)\}\right)$$

$$=\cap\left\{f^{-1}(Y-G):G\in w,V\subseteq(Y-G)\right\}.$$

Since f is (μ,w)-continuous and $G\in w$, $f^{-1}(Y-G)$ is μ-closed in (X,μ) and hence $f^{-1}(c_w(V))$ is μ-closed in (X,μ). On the other hand, $f^{-1}(X-V)$ is μ-closed in (X,μ) and hence $f^{-1}(w\text{-}Fr(V))$ μ-closed in (X,μ). This shows that f is weakly* (μ,w)-continuous.

Conversely, suppose that μ is a topology. Let $x\in X$ and V be a w-open set containing $f(x)$. Then $f(x)\notin w\text{-}Fr(V)$. Since f is weakly (μ,w)-continuous, there exists a μ-open set G containing x such that $f(G)\subseteq c_w(V)$. Put $U=G\cap(X-f^{-1}(w\text{-}Fr(V)))$. Then U is a μ-open set of X containing x and

$$f(U)\subseteq f(G)\cap(Y-w-Fr(V))\subseteq c_w(V)\cap(Y-w\text{-}Fr(V))=V.$$

This shows that f is (μ,w)-continuous.

Theorem 3.8 *Let μ be a GT on X and w be a WS on Y. A function $f:(X,\mu)\to(Y,w)$ is weakly (μ,w)-continuous at x if and only if for each w-open set V containing $f(x)$, $x\in i_\mu(f^{-1}(c_w(V)))$.*

Proof. Let f be weakly (μ,w)-continuous at x and V a w-open set containing $f(x)$. Then, there exists a μ-open set U containing x such that $f(U)\subseteq c_w(V)$. Then we have $x\in U\subseteq f^{-1}(c_w(V))$ and hence $x\in i_\mu(f^{-1}(c_w(V)))$.

Conversely, let V be a w-open set containing $f(x)$. Then, we have $x\in i_\mu(f^{-1}(c_w(V)))$. There exists a μ-open set U such that $x\in U$ and $U\subseteq f^{-1}(c_w(V))$, hence $f(U)\subseteq c_w(V)$. This shows that f is weakly (μ,w)-continuous at x.

Theorem 3.9 *Let μ be a GT on X and w be a WS on Y. A function $f:(X,\mu)\to(Y,w)$ is weakly (μ,w)-continuous if and only if $f^{-1}(V)\subseteq i_\mu(f^{-1}(c_w(V)))$ for every w-open set V of Y.*

Proof. Suppose that f is weakly (μ,w)-continuous. Let $x\in f^{-1}(V)$, then $f(x)\in V$. Since f is weakly (μ,w)-continuous at x, by Theorem 3.8 we have $x\in i_\mu(f^{-1}(c_w(V)))$ and hence $f^{-1}(V)\subseteq i_\mu(f^{-1}(c_w(V)))$.

Conversely, let x be any point of X and $V\in\mu$ containing $f(x)$. Then, we have $x\in f^{-1}(V)\subseteq i_\mu(f^{-1}(c_w(V)))$. By Theorem 3.8, f is weakly (μ,w)-continuous.

We will denote by M_μ the union of all μ-open sets in a GTS (X,μ).

Definition 3.10 *A GTS (X,μ) (resp. WSS (X,w)) is said to be μ-connected (Omari and Noiri (2012)) (resp. w-connected) if M_μ (resp. X) cannot be expressed as the union of two disjoint non-empty μ-open (resp. w-open) subsets of X.*

Theorem 3.11 *Let μ be a GT on X and w be a WS on Y. Let $f:(X,\mu)\to(Y,w)$ be a weakly (μ,w)-continuous surjection. If (X,μ) is μ-connected, then (Y,w) is w-connected.*

Proof. Let $f:(X,\mu)\to(Y,w)$ be a weakly (μ,w)-continuous function of a μ-connected space (X,μ) onto (Y,w). If possible, let Y be w-disconnected. Let V_1 and V_2 form a disconnection of Y. Then V_1 and V_2 are w-open and $Y=V_1\cup V_2$ where $V_1\cap V_2=\phi$. Since V_i is w-open, V_i is w-closed in (Y,w) for each $i=1,2$. Therefore, by Theorem 3.9 we obtain $f^{-1}(V_i)\subseteq i_\mu(f^{-1}(c_w(V_i)))=i_\mu(f^{-1}(V_i))$ for $i=1,2$ and hence $f^{-1}(V_i)$ is μ-open for each $i=1,2$.

Moreover, $M_\mu=f^{-1}(Y)=f^{-1}(V_1\cup V_2)=f^{-1}(V_1)\cup f^{-1}(V_2)$, where $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are non-empty μ-open sets in X. Also $f^{-1}(V_1)\cap f^{-1}(V_2)=\phi$. Hence, X is not μ-connected. This is a contradiction. Therefore, (Y,w) is w-connected.

Corollary 3.12 *Let μ be a GT on X and w be a WS on Y. Let $f:(X,\mu)\to(Y,w)$ be a (μ,w)-continuous surjection. If (X,μ) is μ-connected, then (Y,w) is w-connected.*

4. WEAKLY (w,μ)-CONTINUOUS FUNCTIONS

Definition 4.1 *A function $f:(X,w)\to(Y,\mu)$, where (X,w) is a WSS and (Y,μ) is a GTS, is said to be weakly (w,μ)-continuous if for each $x\in X$ and each $V\in\mu$ containing $f(x)$, there exists $U\in w$ containing x such that $f(U)\subseteq c_\mu(V)$.*

Throughout the present section, (X,w) (resp. (Y,μ)) denotes a WSS (resp. GTS).

Theorem 4.2 *For a function $f:(X,w)\to(Y,\mu)$, the following properties are equivalent:*

1. f is weakly (w,μ)-continuous;
2. $f^{-1}(V)\subseteq i_w(f^{-1}(c_\mu(V)))$ for every μ-open subset V of Y;
3. $c_w(f^{-1}(i_\mu(A)))\subseteq f^{-1}(A)$ for every μ-closed set A of Y;
4. $c_w(f^{-1}(i_\mu(c_\mu(B))))\subseteq f^{-1}(c_\mu(B))$ for every set B of Y;
5. $f^{-1}(i_\mu(B))\subseteq i_w(f^{-1}(c_\mu(i_\mu(B))))$ for every set B of Y;
6. $c_w(f^{-1}(V))\subseteq f^{-1}(c_\mu(V))$ for every μ-open subset V of Y.

Proof. (1) $\Rightarrow$ (2). Let V be a μ-open subset of Y and $x\in f^{-1}(V)$. There exists a w-open set U containing x such that $f(U)\subseteq c_\mu(V)$. Since $x\in U\subseteq f^{-1}(c_\mu(V))$, it follows that $x\in i_w(f^{-1}(c_\mu(V)))$.
Hence $f^{-1}(V)\subseteq i_w(f^{-1}(c_\mu(V)))$.

(2) $\Rightarrow$ (3). Let A be any μ-closed subset of Y.
Then $Y-A$ is μ-open in Y and by (2)
$f^{-1}(Y-A)\subseteq i_w(f^{-1}(c_\mu(Y-A)))=i_w(f^{-1}(Y-i_\mu(A)))=X-c_w(f^{-1}(i_\mu(A)))$.
Therefore, we obtain $c_w(f^{-1}(i_\mu(A)))\subseteq f^{-1}(A)$.

(3) $\Rightarrow$ (4). Let B be a subset of Y. Since $c_\mu(B)$ is μ-closed in Y, by (3) it follows that $c_w(f^{-1}(i_\mu(c_\mu(B))))\subseteq f^{-1}(c_\mu(B))$.

(4) $\Rightarrow$ (5). Let B be a subset of Y. Form (4), it follows

$$f^{-1}(i_\mu(B))=X-f^{-1}(c_\mu(Y-B))\subseteq X-c_w(f^{-1}(i_\mu(c_\mu(Y-B))))$$
$$=i_w(f^{-1}(c_\mu(i_\mu(B)))).$$

Therefore, we obtain $f^{-1}(i_\mu(B))\subseteq i_w(f^{-1}(c_\mu(i_\mu(B))))$.

(5) $\Rightarrow$ (6). Let V be any μ-open subset of Y. Suppose $x\notin f^{-1}(c_\mu(V))$. Then $f(x)\notin c_\mu(V)$ and hence there exists a μ-open set U containing $f(x)$ such that $U\cap V=\phi$ and hence $c_\mu(U)\cap V=\phi$. By (5), $x\in f^{-1}(U)\subseteq i_w(f^{-1}(c_\mu(U)))$ and hence there exists a w-open set G containing x such that $x\in G\subseteq f^{-1}(c_\mu(U))$. Since $c_\mu(U)\cap V=\phi$ and

$f(G) \subseteq c_\mu(U)$, we have $G \cap f^{-1}(V) = \phi$ and hence $x \notin c_w(f^{-1}(V))$. Hence $c_w(f^{-1}(V)) \subseteq f^{-1}(c_\mu(V))$.

(6) $\Rightarrow$ (1). Let $x \in X$ and $V \in \mu$ containing $f(x)$. From (6),

$$x \in f^{-1}(V) \subseteq f^{-1}(i_\mu(c_\mu(V))) = X - f^{-1}(c_\mu(Y - c_\mu(V)))$$
$$\subseteq X - c_w(f^{-1}(Y - c_\mu(V))) = i_w(f^{-1}(c_\mu(V))).$$

So there exists a w -open subset U containing x in X such that $U \subseteq f^{-1}(c_\mu(V))$. Hence $f(U) \subseteq c_\mu(V)$ and f is weakly (w, μ) -continuous.

Theorem 4.3 *Let $f:(X,w) \to (Y,\mu)$ be a function. The set of all points $x \in X$ at which f is not weakly (w,μ) -continuous is identical with the union of w-frontier of the inverse images of the μ-closure of μ-open sets containing $f(x)$.*

Proof. Suppose that f is not weakly (w,μ) -continuous at $x \in X$. There exists a μ -open sets V of Y containing $f(x)$ such that $f(U)$ is not contained in $c_\mu(V)$ for every $U \in w$ containing x. Then $U \cap (X - f^{-1}(c_\mu(V))) \neq \phi$ for every $U \in w$ containing x and hence $x \in c_w(X - f^{-1}(c_\mu(V)))$. On the other hand, we have $x \in f^{-1}(V) \subseteq c_w(f^{-1}(c_\mu(V)))$ and hence $x \in w\text{-}Fr(f^{-1}(c_\mu(V)))$.

Conversely, suppose that f is weakly (μ, w) -continuous at $x \in X$ and let V be any μ -open set of Y containing $f(x)$. Then, by Theorem 4.2 we have $x \in f^{-1}(V) \subseteq i_w(f^{-1}(c_\mu(V))$. Therefore, $x \notin w\text{ - }Fr(f^{-1}(c_\mu(V)))$ for each μ -open set V of Y containing $f(x)$. This completes the proof.

Definition 4.4 Császár (2002). *Let w be a WS on X and $A \subseteq X$. Then*

1. $A \in \alpha(w)$ if $A \subseteq i_w(c_w(i_w(A)))$.
2. $A \in \sigma(w)$ if $A \subseteq c_w(i_w(A))$.

3. $A \in \pi(w)$ if $A \subseteq i_w(c_w(A))$.
4. $A \in \beta(w)$ if $A \subseteq c_w(i_w(c_w(A)))$.

Lemma 4.5 Császár (2002). *If w is a WS, we have*

1. $w \subseteq \alpha(w) \subseteq \sigma(w) \subseteq \beta(w)$..
2. $w \subseteq \alpha(w) \subseteq \pi(w) \subseteq \beta(w)$..
3. each of $\alpha(w)$, $\pi(w)$, $\sigma(w)$ and $\beta(w)$ is a generalized topology.

Remark 4.6 *Definition 4.4 and Lemma 4.5, in case $w = \mu$ (generalized topology), are found in the paper (Császár (2002)) due to Császár.*

Definition 4.7 Ekici (2012). *Let w be a WS on X and $A \subseteq X$. Then*

1. $A \in r(w)$ if $A = i_w(c_w(A))$.
2. $A \in rc(w)$ if $A = c_w(i_w(A))$.

Theorem 4.8 Ekici (2012). *The following properties are equivalent for a WS w on X and $A \subseteq X$.*

1. $A \in \beta(w)$.
2. there exists $B \in \pi(w)$ such that $B \subseteq c_w(A) \subseteq c_w(B)$.
3. $c_w(A) \in rc(w)$.

Definition 4.9 *Let μ be a GT on X and $A \subseteq X$. Then*

1. $A \in r(\mu)$ if $A = i_\mu(c_\mu(A))$.
2. $A \in rc(\mu)$ if $A = c_\mu(i_\mu(A))$.

Corollary 4.10 *The following properties are equivalent for a GT μ on X and $A \subseteq X$.*

1. $A \in \beta(\mu)$,
2. there exists $B \in \pi(\mu)$ such that $B \subseteq c_\mu(A) \subseteq c_\mu(B)$,
3. $c_\mu(A) \in rc(\mu)$.

Lemma 4.11 *If w is a WS, then $w \subseteq w^* \subseteq \alpha(w)$.*

Theorem 4.12 *For a function $f:(X,w) \to (Y,\mu)$, the following properties are equivalent:*

1. f is weakly (w,μ)-continuous;
2. $c_w(f^{-1}(i_\mu(A))) \subseteq f^{-1}(A)$ for every $A \in rc(\mu)$;
3. $c_w(f^{-1}(i_\mu(c_\mu(G)))) \subseteq f^{-1}(c_\mu(G))$ for every $G \in \beta(\mu)$;
4. $c_w(f^{-1}(i_\mu(c_\mu(G)))) \subseteq f^{-1}(c_\mu(G))$ for every $G \in \sigma(\mu)$.

Proof. (1) $\Rightarrow$ (2). Let $A \in rc(\mu)$. Since $i_\mu(A) \in \mu$, by Theorem 4.2 (6) and $A \in rc(\mu)$, $c_w(f^{-1}(i_\mu(A))) \subseteq f^{-1}(c_\mu(i_\mu(A))) = f^{-1}(A)$.

(2) $\Rightarrow$ (3). Let $G \in \beta(\mu)$ then by Corollary 4.10 , $c_\mu(G) \in rc(\mu)$. Form (2), it follows $c_w(f^{-1}(i_\mu(c_\mu(G)))) \subseteq f^{-1}(c_\mu(G))$.

(3) $\Rightarrow$ (4). Since $\sigma(\mu) \subseteq \beta(\mu)$, the proof is obvious.

(4) $\Rightarrow$ (1). Let $V \in \mu$. Then since $\mu \subseteq \sigma(\mu)$, by (4) $c_w(f^{-1}(V)) \subseteq c_w(f^{-1}(i_\mu(c_\mu(V)))) \subseteq f^{-1}(c_\mu(V))$. Hence by Theorem 4.2 (6), f is weakly (w,μ)-continuous.

Theorem 4.13 *For a function $f:(X,w) \to (Y,\mu)$, the following properties are equivalent:*

1. f is weakly (w,μ)-continuous;
2. $c_w(f^{-1}(i_\mu(c_\mu(G)))) \subseteq f^{-1}(c_\mu(G))$ for every $G \in \pi(\mu)$;
3. $c_w(f^{-1}(G)) \subseteq f^{-1}(c_\mu(G))$ for every $G \in \pi(\mu)$;
4. $f^{-1}(G) \subseteq i_w(f^{-1}(c_\mu(G)))$ for every $G \in \pi(\mu)$.

Proof. (1) $\Rightarrow$ (2). Let $G \in \pi(\mu)$. Then $c_\mu(G) = c_\mu(i_\mu(c_\mu(G)))$ and $c_\mu(G) \in rc(\mu)$.

Form Theorem 4.12, it follows that $c_w(f^{-1}(i_\mu(c_\mu(G)))) \subseteq f^{-1}(c_\mu(G))$.

(2) $\Rightarrow$ (3). The proof is obvious.

(3) $\Rightarrow$ (4). Let $G \in \pi(\mu)$. Then it follows from (3) that

$$f^{-1}(G) \subseteq f^{-1}(i_\mu(c_\mu(G))) = X - f^{-1}(c_\mu(Y - c_\mu(G)))$$
$$\subseteq X - c_w(f^{-1}(Y - c_\mu(G))) = i_w(f^{-1}(c_\mu(G))).$$

Hence we have (4).

(4) $\Rightarrow$ (1). Since $\mu \subseteq \pi(\mu)$, from (4) and Theorem 4.2, it follows that weakly (w,μ)-continuous.

Theorem 4.14 *For a function $f:(X,w) \to (Y,\mu)$, the following properties are equivalent:*

1. f is weakly (w,μ)-continuous;
2. $c_w(f^{-1}(i_\mu(\gamma_\theta(B)))) \subseteq f^{-1}(\gamma_\theta(B))$ for every subset B of Y ;
3. $c_w(f^{-1}(i_\mu(c_\mu(B)))) \subseteq f^{-1}(\gamma_\theta(B))$ for every subset B of Y ;
4. $c_w(f^{-1}(i_\mu(c_\mu(G)))) \subseteq f^{-1}(c_\mu(G))$ for every μ-open set G of Y ;
5. $c_w(f^{-1}(i_\mu(c_\mu(V)))) \subseteq f^{-1}(c_\mu(V))$ for every $V \in \pi(\mu)$;
6. $c_w(f^{-1}(i_\mu(K))) \subseteq f^{-1}(K)$ for every $K \in rc(\mu)$;
7. $c_w(f^{-1}(i_\mu(c_\mu(G)))) \subseteq f^{-1}(c_\mu(G))$ for every $G \in \beta(\mu)$;
8. $c_w(f^{-1}(i_\mu(c_\mu(G)))) \subseteq f^{-1}(c_\mu(G))$ for every $G \in \sigma(\mu)$;
9. $f(c_w(A)) \subseteq \gamma_\theta(f(A))$ for every subset A of X ;
10. $c_w(f^{-1}(B)) \subseteq f^{-1}(\gamma_\theta(B))$ for every subset B of Y.

Proof. (1) $\Rightarrow$ (2). Let B be any subset of Y. Then by Lemma 1.5 $\gamma_\theta(B)$ is μ-closed in Y. Then by Theorem 4.2 $c_w(f^{-1}(i_\mu(\gamma_\theta(B)))) \subseteq f^{-1}(\gamma_\theta(B))$.

(2) $\Rightarrow$ (3). This is obvious since $c_\mu(B) \subseteq \gamma_\theta(B)$ for every subset B.

(3) $\Rightarrow$ (4). This is obvious since $c_\mu(G) = \gamma_\theta(G)$ for every μ-open set G.

(4) $\Rightarrow$ (5). Let $V \in \pi(\mu)$. Then we have $V \subseteq i_\mu(c_\mu(V))$ and

$c_\mu(V)=c_\mu(i_\mu(c_\mu(V)))$. Now, set $G=i_\mu(c_\mu(V))$ then G is μ-open in Y and $c_\mu(V)=c_\mu(G)$.

Therefore, by (4) we have $c_w(f^{-1}(i_\mu(c_\mu(V))))\subseteq f^{-1}(c_\mu(V))$.

(5) $\Rightarrow$ (6). Let $K\in rc(\mu)$. Then we have $i_\mu(K)\in\pi(\mu)$ and hence by (5) $c_w(f^{-1}(i_\mu(K)))=c_w(f^{-1}(i_\mu(c_\mu(i_\mu(K)))))\subseteq f^{-1}(c_\mu(i_\mu(K)))=f^{-1}(K)$.

(6) $\Rightarrow$ (7). Let $G\in\beta(\mu)$. Then $G\subseteq c_\mu(i_\mu(c_\mu(G)))$. Since $c_\mu(G)\in rc(\mu)$, by (6) $c_w(f^{-1}(i_\mu(c_\mu(G))))\subseteq f^{-1}(c_\mu(G))$.

(7) $\Rightarrow$ (8). This is obvious since $\sigma(\mu)\subseteq\beta(\mu)$.

(8) $\Rightarrow$ (1). Let V be any μ-open set of Y. Then, by (8) we have $c_w(f^{-1}(V))\subseteq c_w(f^{-1}(i_\mu(c_\mu(V))))\subseteq f^{-1}(c_\mu(V))$. It follows from Theorem 4.2 that f is weakly (w,μ)-continuous.

(1) $\Rightarrow$ (9). Let A be any subset of X. Let $x\in c_w(A)$ and V be any μ-open set of Y containing $f(x)$. There exists $U\in w$ containing x such that $f(U)\subseteq c_\mu(V)$. Since $x\in c_w(A)$, then we have $U\cap A\neq\phi$ and hence $\phi\neq f(U)\cap f(A)\subseteq c_\mu(V)\cap f(A)$. Therefore, we have $f(x)\in\gamma_\theta(f(A))$ and hence $f(c_w(A))\subseteq\gamma_\theta(f(A))$.

(9) $\Rightarrow$ (10). Let B be any subset of Y. By (9), we have $f(c_w(f^{-1}(B)))\subseteq\gamma_\theta(f(f^{-1}(B)))\subseteq\gamma_\theta(B)$ and hence $c_w(f^{-1}(B))\subseteq f^{-1}(\gamma_\theta(B))$.

(10) $\Rightarrow$ (1). Let B be any subset of Y. By (10), we have
$c_w(f^{-1}(i_\mu(c_\mu(B))))\subseteq f^{-1}(\gamma_\theta(i_\mu(c_\mu(B))))=f^{-1}(c_\mu(i_\mu(c_\mu(B))))\subseteq f^{-1}(c_\mu(B))$.

It follows from Theorem 4.2 that f is weakly (w,μ)-continuous.

CONCLUSION

In this paper, we obtained many characterizations of the following functions:

1. a (w,k)-continuous function $f:(X,w)\to(Y,k)$, where w and k are weak structures,
2. a (μ,k) -continuous function $f:(X,\mu)\to(Y,w)$, where μ is a generalized topology and w is a weak structure,
3. a weakly (μ, w) -continuous function $f:(X,\mu)\to(Y,w)$, where μ is a generalized topology and w is a weak structure,
4. a weakly (w,μ) -continuous function $f:(X,w)\to(Y,\mu)$, where w is a weak structure and μ is a generalized topology.

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